

Exploring Analytic Geometry with *Mathematica*[®]

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Preface

The study of two-dimensional analytic geometry has gone in and out of fashion several times over the past century, however this classic field of mathematics has once again become popular due to the growing power of personal computers and the availability of powerful mathematical software systems, such as *Mathematica*, that can provide an interactive environment for studying the field. By combining the power of *Mathematica* with an analytic geometry software system called *Descarta2D*, the author has succeeded in meshing an ancient field of study with modern computational tools, the result being a simple, yet powerful, approach to studying analytic geometry. Students, engineers and mathematicians alike who are interested in analytic geometry can use this book and software for the study, research or just plain enjoyment of analytic geometry.

Mathematica provides an attractive environment for studying analytic geometry. *Mathematica* supports both numeric and symbolic computations, meaning that geometry problems can be solved numerically, producing approximate or exact answers, as well as producing general formulas with variables. *Mathematica* also has good facilities for producing graphical plots which are useful for visualizing the graphs of two-dimensional geometry.

Features

Exploring Analytic Geometry with Mathematica, *Mathematica* and *Descarta2D* provide the following outstanding features:

- The book can serve as classical analytic geometry textbook with in-line *Mathematica* dialogs to illustrate key concepts.
- A large number of examples with solutions and graphics is keyed to the textual development of each topic.
- Hints are provided for improving the reader's use and understanding of *Mathematica* and *Descarta2D*.
- More advanced topics are covered in explorations provided with each chapter, and full solutions are illustrated using *Mathematica*.

- A detailed reference manual provides complete documentation for *Descarta2D*, with complete syntax for over 100 new commands.
- Complete source code for *Descarta2D* is provided in 30 well-documented *Mathematica* notebooks.
- The complete book is integrated into the *Mathematica* Help Browser for easy access and reading.
- A CD-ROM is included for convenient, permanent storage of the *Descarta2D* software.
- A complete software system and mathematical reference is packaged as an affordable book.

Classical Analytic Geometry

Exploring Analytic Geometry with Mathematica begins with a traditional development of analytic geometry that has been modernized with in-line chapter dialogs using *Descarta2D* and *Mathematica* to illustrate the underlying concepts. The following topics are covered in 21 chapters:

Coordinates • Points • Equations • Graphs • Lines • Line Segments • Circles • Arcs • Triangles • Parabolas • Ellipses • Hyperbolas • General Conics • Conic Arcs • Medial Curves • Transformations • Arc Length • Area • Tangent Lines • Tangent Circles • Tangent Conics • Biarcs.

Each chapter begins with definitions of underlying mathematical terminology and develops the topic with more detailed derivations and proofs of important concepts.

Explorations

Each chapter in *Exploring Analytic Geometry with Mathematica* concludes with more advanced topics in the form of exploration problems to more fully develop the topics presented in each chapter. There are more than 100 of these more challenging explorations, and the full solutions are provided on the CD-ROM as *Mathematica* notebooks as well as printed in Part VIII of the book. Sample explorations include some of the more famous theorems from analytic geometry:

Carlyle's Circle • Castillon's Problem • Euler's Triangle Formula • Eyeball Theorem • Gergonne's Point • Heron's Formula • Inversion • Monge's Theorem • Reciprocal Polars • Reflection in a Point • Stewart's Theorem • plus many more.

Descarta2D

Descarta2D provides a full-scale *Mathematica* implementation of the concepts developed in *Exploring Analytic Geometry with Mathematica*. A reference manual section explains in detail the usage of over 100 new commands that are provided by *Descarta2D* for creating, manipulating and querying geometric objects in *Mathematica*. To support the study and enhancement of the *Descarta2D* algorithms, the complete source code for *Descarta2D* is provided, both in printed form in the book and as *Mathematica* notebook files on the CD-ROM.

CD-ROM

The CD-ROM provides the complete text of the book in Abode Portable Document Format (PDF) for interactive reading. In addition, the CD-ROM provides the following *Mathematica* notebooks:

- Chapters with *Mathematica* dialogs, 24 interactive notebooks
- Reference material for *Descarta2D*, three notebooks
- Complete *Descarta2D* source code, 30 notebooks
- *Descarta2D* packages, 30 loadable files
- Exploration solutions, 125 notebooks.

These notebooks have been thoroughly tested and are compatible with *Mathematica* Version 3.0.1 and Version 4.0. Maximum benefit of the book and software is gained by using it in conjunction with *Mathematica*, but a passive reading and viewing of the book and notebook files can be accomplished without using *Mathematica* itself.

Organization of the Book

Exploring Analytic Geometry with Mathematica is a 900-page volume divided into nine parts:

- Introduction (Getting Started and *Descarta2D* Tour)
- Elementary Geometry (Points, Lines, Circles, Arcs, Triangles)
- Conics (Parabolas, Ellipses, Hyperbolas, Conics, Medial Curves)
- Geometric Functions (Transformations, Arc Length, Area)
- Tangent Curves (Lines, Circles, Conics, Biarcs)
- *Descarta2D* Reference (philosophy and command descriptions)
- *Descarta2D* Packages (complete source code)

- Explorations (solution notebooks)
- Epilogue (Installation Instructions, Bibliography and a detailed index).

About the Author

Donald L. Vossler is a mechanical engineer and computer software designer with more than 20 years experience in computer aided design and geometric modeling. He has been involved in solid modeling since its inception in the early 1980's and has contributed to the theoretical foundation of the subject through several published papers. He has managed the development of a number of commercial computer aided design systems and holds a US Patent involving the underlying data representations of geometric models.

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The study of two-dimensional analytic geometry has gone in and out of fashion several times over the past century. This classic field of mathematics has once again become popular due to the growing power of personal computers and the availability of powerful mathematical software systems, such as *Mathematica*, that can provide an interactive environment for studying the field.

By combining the power of *Mathematica* with an analytic geometry software system called *Descarta2D*, the author has succeeded in meshing an ancient field of study with modern computational tools, the result being a simple, yet powerful, approach to studying analytic geometry. Students, engineers and mathematicians alike who are interested in analytic geometry can use this book and software for the study, research or just plain enjoyment of analytic geometry.

- A classic study in analytic geometry, complete with in-line *Mathematica* dialogs illustrating every concept as it is introduced.
 - Excellent theoretical presentation
 - Fully explained examples of all key concepts
- Interactive *Mathematica* notebooks for the entire book.
 - provides a complete computer-based environment for study of analytic geometry
 - all chapters and reference material are provided on the CD in addition to being printed in the book.
- Complete software system: *Descarta2D*
 - a software system, including source code, for the underlying computer implementation, called *Descarta2D* is provided
 - Part VII of the book is a listing of the (30) *Mathematica* files notebooks supporting *Descarta2D*; the source code is also in on the CD
- Explorations
 - More than 120 challenging problems in analytic geometry are posed. Complete solutions are provided both as interactive *Mathematica* notebooks on the CD and as printed material in the book.
- *Mathematica* and *Descarta2D* Hints are provided to expand the reader's knowledge and understanding of *Descarta2D* and *Mathematica* .
- Detailed reference manual
 - Complete documentation for *Descarta2D*
 - Fully integrated into the *Mathematica* Help Browser

About the author

Donald L. Vossler is a mechanical engineer and computer software designer with more than 20 years experience in computer aided design and geometric modeling. He has been involved in solid modeling since its inception in the early 1980's and has contributed to the theoretical foundation of the subject through several published papers. He has managed the development of a number of commercial computer aided design systems and holds a US Patent involving the underlying data representations of geometric models.

CD-ROM included

Full contents of book included on CD-ROM, which will operate on Macintosh, Windows and UNIX machines with *Mathematica* 3.0.1 or 4.0 installed.